

Strong solutions to the Navier–Stokes–Fourier system with slip–inflow boundary conditions

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Abstract

We consider a system of partial differential equations describing the steady flow of a compressible heat conducting Newtonian fluid in a three-dimensional channel with inflow and outflow part. We show the existence of a strong solution provided the data are close to a constant, but nontrivial flow with sufficiently large dissipation in the energy equation.

1 Introduction

We investigate the stationary flow of a heat conducting compressible fluid in a cylindrical domain. The fluid is assumed to be Newtonian. Then the flow is described by the stationary Navier–Stokes–Fourier (NSF) system:

$$\begin{aligned}
 \rho v \cdot \nabla v - \operatorname{div} \mathbf{S}(\nabla v) + \nabla \pi(\rho, \theta) &= \rho f && \text{in } \Omega, \\
 \operatorname{div}(\rho v) &= 0 && \text{in } \Omega, \\
 \operatorname{div}(\rho E v) &= \rho f \cdot v - \operatorname{div}(\pi v) + \operatorname{div}(\mathbf{S}(v)v) - \operatorname{div} q && \text{in } \Omega, \\
 (\mathbf{S}(\nabla v)n) \cdot \tau_k + \alpha v \cdot \tau_k &= b_k, \quad k = 1, 2 && \text{on } \Gamma, \\
 n \cdot v &= d && \text{on } \Gamma, \\
 \rho &= \rho_{in} && \text{on } \Gamma_{in}, \\
 -q \cdot n + L(\theta - T) &= g && \text{on } \Gamma,
 \end{aligned} \tag{1.1}$$

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where $\Omega = \Omega_0 \times [0, l]$ with $\Omega_0 \subset \mathbb{R}^2$ smooth, v is the velocity field of the fluid, ρ is the density, θ is the absolute temperature. We assume that the pressure $\pi(\rho, \theta)$ is a twice continuously differentiable function on $\mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\begin{aligned}\pi(1, T_0) &= p_0 > 0, \\ \partial_\rho \pi(1, T_0) &= p_1 > 0, \\ \partial_\theta \pi(1, T_0) &= p_2 > 0.\end{aligned}\tag{1.2}$$

We consider the Newtonian compressible fluid, i.e. the stress tensor has the form

$$\mathbf{S}(\nabla v) = \mu \left(\nabla v + \nabla^T v - \frac{2}{3} \operatorname{div} v \mathbf{I} \right) + \lambda \operatorname{div} v \mathbf{I},\tag{1.3}$$

where the constant viscosities μ and λ fulfill $\mu > 0$ and $\lambda \geq 0$.

Further, the friction coefficient $\alpha > 0$. Finally, $q = -\kappa \nabla \theta$ is the heat flux. Note that we could treat the situation $\kappa = \kappa(\rho, \theta)$, $\mu = \mu(\rho, \theta)$ and $\lambda = \lambda(\rho, \theta)$ with suitable assumptions on these functions. It would only lead to further complications, therefore we take them rather constant not to hide the main ideas by too many technicalities.

The boundary of the domain is divided in a natural way into the inflow part Γ_{in} , the outflow part Γ_{out} and the impermeable wall Γ_0 . More precisely, we define

$$\begin{aligned}\Gamma_{in} &= \{x \in \Gamma : d < 0\}, \\ \Gamma_{out} &= \{x \in \Gamma : d > 0\}, \\ \Gamma_0 &= \{x \in \Gamma : d = 0\}.\end{aligned}$$

We consider a channel-like flow, i.e. we further assume that $\Gamma_{in} = \Omega_0 \times \{0\}$, $\Gamma_{out} = \Omega_0 \times \{l\}$ and $\Gamma_0 = \partial\Omega_0 \times [0, l]$. We set $L = 0$ on $\Gamma_{in} \cup \Gamma_{out}$ and L to be a given positive constant on the wall Γ_0 . We set $g = 0$ on Γ_0 but we admit $g \neq 0$ on $\Gamma_{in} \cup \Gamma_{out}$ (see fig.1). Under such

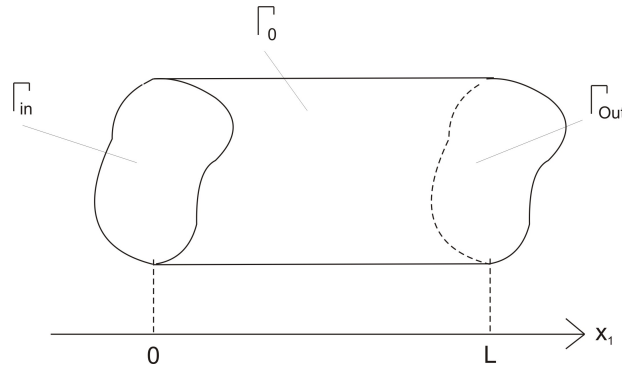


Figure 1: The domain

choice, on the inflow and outflow part the boundary condition for the temperature reduces to the inhomogeneous Neumann condition. From the point of view of modeling these parts of the boundary are artificial and we can measure the parameters of the flow, what can be reflected in

different values of g . The condition on Γ_0 means that the flux of the temperature through the wall is proportional to the difference between the outside temperature T and the temperature of the fluid. The level of thermal insulation of the wall is given by L . Existence and uniqueness of the special solution $\bar{\theta}$ under our choice of boundary conditions is discussed in the next chapter.

The total energy is given by $E = \frac{1}{2}|u|^2 + e(\rho, \theta)$, where the internal energy e is given by

$$e(\rho, \theta) = e_\pi(\rho, \theta) + c_v \theta \quad (1.4)$$

with

$$\partial_\rho e_\pi(\rho, \theta) = \frac{1}{\rho^2} (\pi(\rho, \theta) - \theta \partial_\theta \pi(\rho, \theta)), \quad (1.5)$$

i.e. the internal energy fulfills the Maxwell relation. Without loss of generality we set in (1.4) $c_v = 1$. Equivalently we can consider system (1.1) with total energy balance (1.1)₃ replaced with the internal energy balance

$$\operatorname{div}(\rho e v) - \kappa \Delta \theta = \mathbf{S}(\nabla v) : \nabla v - \pi \operatorname{div} v. \quad (1.6)$$

We also set

$$\partial_\rho e_\pi(1, T_0) = e_1, \quad \partial_\theta e_\pi(1, T_0) = e_2. \quad (1.7)$$

Known results

The existence theory for the steady compressible Navier–Stokes equations in the framework of strong solutions was intensively studied in eighties, see e.g. [20], [21] and many other papers. In the beginning of this century, inspired by the results of P.L. Lions (see [13]), the attention turned rather towards the weak solutions, i.e. towards the existence theory without any smallness assumption on the data. The best results in this direction so far can be found in [8] (slip boundary conditions) and [10] (space periodic situation). Note, however, that the theory is not able to treat the situation with nonzero inflow/outflow conditions.

Strong solutions with inhomogeneous boundary data, which are in our scope of interest in this paper, have been considered for the first time in [29] in Hilbert spaces and later in L_p framework ([11], [12], [24]).

In [23] a constant flow in the direction of the axis of the cylinder is investigated and existence of a strong solutions in its vicinity is shown in the barotropic case.

Concerning the NSF system, the existence of weak solutions was studied much later, see [17], [18]. The best result in this direction can be found in [9]. Note, however, that similarly to the barotropic case, all these results do not allow to treat the inflow/outflow problems.

The theory of strong solutions for the NSF system describing the thermal effects is much less developed, to our knowledge the only result in L_p framework is the paper by Beirao da Veiga [4] where existence of strong solutions in a vicinity of a constant flow with zero velocity is shown.

Special solution. Here we are interested in showing existence of strong solutions, i.e. we are looking for $(v, \rho, \theta) \in W_p^2(\Omega) \times W_p^1(\Omega) \times W_p^2(\Omega)$, solving system (1.1) so that the solution is close to a given special solution. Hence we arrive at a problem with some smallness of the data. Even though smallness obviously facilitates the analysis, the results of this kind are still missing in the theory of compressible flow, especially for heat-conducting fluids (see the overview

above). This is mostly due to high complexity of the system $(1.1)_{1-3}$, in particular due to its mixed, elliptic (or parabolic in time dependent case) and hyperbolic character. Investigation of stationary solutions close to given laminar flows is of particular importance if we want to investigate stability of special solutions, which is quite well developed for weak solutions but so far almost not investigated in strong solutions framework (see [22] for an overview of results).

The simplest example of a special solution to perturb is a solution with zero velocity and constant density and temperature. Such flow has been investigated in [4] with homogeneous Dirichlet boundary conditions. From the point of view of applications it is important to investigate flows with large velocity, which leads in a natural way to inhomogeneous boundary data. In a cylindrical domain a natural example of such solution is a constant flow in the direction of the axis of the cylinder:

$$\bar{v} = [1, 0, 0], \quad \bar{\rho} = 1. \quad (1.8)$$

In other words, we can state that such a flow is natural to be investigated as a special solution to the system (1.1) in our domain Ω

As for the temperature of the basic solution the simplest choice would be to assume it constant. It seems however quite artificial with the flow $(\bar{v}, \bar{\rho})$ defined above. It is more interesting from physical point of view to consider a flow with nontrivial temperature function $\bar{\theta}$ which corresponds to the solution $(\bar{v}, \bar{\rho})$ as a solution to corresponding energy balance equation. On the boundary we assume the temperature of the form $T = T_0 + T_1$ with constant, possibly large T_0 and small variation T_1 . Substituting $(\bar{v}, \bar{\rho})$ to the internal energy balance equation we see that $\bar{\theta}$ should satisfy the system

$$\begin{aligned} (1 + e_2)\partial_{x_1}\bar{\theta} - \kappa\Delta\bar{\theta} &= 0 & \text{in } \Omega, \\ \kappa\frac{\partial\bar{\theta}}{\partial n} + L(\bar{\theta} - T_0 - T_1) &= g & \text{on } \Gamma. \end{aligned} \quad (1.9)$$

Now we write $\bar{\theta} = \bar{\theta}_1 + \bar{\theta}_2 = T_0 + (\bar{\theta}_0 - T_0) + \bar{\theta}_1$, where $\bar{\theta}_0$ solves

$$\begin{aligned} -\kappa\Delta\bar{\theta}_0 &= c_g & \text{in } \Omega, \\ \kappa\frac{\partial\bar{\theta}_0}{\partial n}|_{\Gamma} &= g & \text{on } \Gamma, \\ \frac{1}{|\Omega|} \int_{\Omega} \bar{\theta}_0 \, dx &= T_0, \end{aligned} \quad (1.10)$$

and

$$c_g = \frac{1}{|\Omega|} \int_{\Gamma} g \, dS. \quad (1.11)$$

The existence of a solution $\bar{\theta}_0 \in W_p^2$ to (1.10) is a classical elliptic result, we only need to apply symmetry to deal with corner singularities. The details are given in Lemma 6. In particular $\bar{\theta}_0$ satisfies

$$\|\nabla^2\bar{\theta}_0\|_{L_p} + \|\nabla\bar{\theta}_0\|_{L_p} + \|\bar{\theta}_0 - T_0\|_{L_p} \leq C[\|g\|_{W_p^{1-1/p}(\Gamma)} + \|c_g\|_{L_p(\Gamma)}] \leq C\|g\|_{W_p^{1-1/p}(\Gamma)}. \quad (1.12)$$

Now we see that $\bar{\theta}_1$ satisfies

$$\begin{aligned} (1 + e_2)\partial_{x_1}\bar{\theta}_1 - \kappa\Delta\bar{\theta}_1 &= -(1 + e_2)\partial_{x_1}\bar{\theta}_0 - c_g & \text{in } \Omega, \\ \kappa\frac{\partial\bar{\theta}_1}{\partial n} + L(\bar{\theta}_1 - (T_1 + T_0 - \bar{\theta}_0)) &= 0 & \text{on } \Gamma. \end{aligned} \quad (1.13)$$

The existence and uniqueness of solution to the above system is discussed in Subsection 2.2.

Our goal is to show the existence of a solution to (1.1) close to $(\bar{v}, \bar{\rho}, \bar{\theta})$. Hence it is convenient to introduce the following quantity to measure the distance of the data from the special solution:

$$\begin{aligned} D_0 &= \|f\|_{L_p(\Omega)} + \sum_{i=1}^2 \|b_i - \alpha \tau_i^{(1)}\|_{W_p^{1-1/p}(\Gamma)} + \|d - n^{(1)}\|_{W_p^{2-1/p}(\Gamma)} \\ &+ \|\rho_{in} - 1\|_{W_p^1(\Gamma_{in})} + \|g\|_{W_p^{1-1/p}(\Gamma)} + \|T_1\|_{W_p^{1-1/p}(\Gamma)}. \end{aligned} \quad (1.14)$$

We are now in a position to formulate our main result.

Theorem 1. *Assume that D_0 defined in (1.14) is small enough, κ is large enough, L is large enough on Γ_0 , α large enough on Γ_{in} and $p > 3$. Then there exists a solution (v, ρ, θ) to system (1.1) such that*

$$\|v - \bar{v}\|_{W_p^2} + \|\rho - \bar{\rho}\|_{W_p^1} + \|\theta - \bar{\theta}\|_{W_p^2} \leq E(D_0). \quad (1.15)$$

This solution is unique in the class of solutions satisfying (1.15).

Notation. We use standard notation for the Sobolev spaces, i.e. $W_p^1(\Omega)$. For the spaces defined on Ω we will skip the domain; for example we write L_2 instead of $L_2(\Omega)$. A generic constant that is controlled (but not necessarily small) will be denoted by C , whereas E will denote a small constant (i.e. which can be arbitrarily small for the data small enough). We will also use the space $L_\infty((0, L), L_2(\Omega_0))$, for simplicity we denote it by $L_\infty(L_2)$.

2 Preliminaries

2.1 Auxiliary results

In this section we collect some standard results which we use throughout the paper. We start with the Sobolev imbedding theorem, let us state here the particular case which is used throughout the paper (the general version with the proof can be found in [1, Theorem 5.4])

Lemma 1. *Let Ω be defined as above and let $f \in W_p^1(\Omega)$ with $p > 3$. Then $f \in L_\infty(\Omega)$ and there exist $C = C(p, \Omega)$ s.t.*

$$\|f\|_{L_\infty} \leq C \|f\|_{W_p^1}.$$

Next we recall the Poincaré inequality. Among its different versions we will need the following two.

Lemma 2. *Let Γ_1 and Γ_2 denote two nontrivial parts of the boundary $\Gamma = \partial\Omega$ and $u \in W_2^1(\Omega)$. Then*

$$\|u\|_{L_2(\Gamma_1)}^2 \leq C_P [\|u\|_{L_2(\Gamma_2)}^2 + \|\nabla u\|_{L_2(\Omega)}^2] \quad (2.1)$$

and

$$\|u\|_{L_2(\Omega)}^2 \leq C_P [\|u\|_{L_2(\Gamma_1)}^2 + \|\nabla u\|_{L_2(\Omega)}^2], \quad (2.2)$$

where $C_P = C_P(\Omega)$.

Lemma 3. (interpolation inequality):

Let $2 < p < \infty$. Then for all $\epsilon > 0$ $\exists C(\epsilon, p, \Omega)$ such that $\forall f \in W_p^1(\Omega)$:

$$\|f\|_{L_p} \leq \epsilon \|\nabla f\|_{L_p} + C \|f\|_{L_2}. \quad (2.3)$$

Proof. The interpolation inequality in the Lebesgue spaces and the imbedding $W_p^1 \hookrightarrow L_q$ with $q = \infty$ for $p > 3$, $q < \infty$, arbitrary for $p = 3$ and $q = \frac{3p}{3-p}$ for $p < 3$ yield

$$\|f\|_{L_p} \leq C(p) \|f\|_{L_q}^\theta \|f\|_{L_2}^{1-\theta} \leq C(p) (\|f\|_{L_p} + \|\nabla f\|_{L_p})^\theta \|f\|_{L_2}^{1-\theta}.$$

Now application of the Young inequality leads to (2.3). \square

Another result we use is the following version of the Korn inequality

Lemma 4. Let $\Omega \in C^{0,1}$, $\mu > 0$ and if $\alpha = 0$ then Ω is not axially symmetric. Then

$$\int_{\Omega} \mu |\nabla u + \nabla^T u - \frac{2}{3} \operatorname{div} u \mathbf{I}|^2 dx + \int_{\Gamma} (\alpha(u \cdot \tau)^2 + d^2) dS \geq C \|u\|_{W_2^1}^2. \quad (2.4)$$

Proof. It is a straightforward generalization of [8, Lemma 4]. \square

We also apply the following well-known result in finite dimensional Hilbert spaces (the proof can be found in [28]):

Lemma 5. Let X be a finite dimensional Hilbert space and let $P: X \rightarrow X$ be a continuous operator satisfying

$$\exists M > 0 : \quad (P(\xi), \xi) > 0 \quad \text{for} \quad \|\xi\| = M. \quad (2.5)$$

Then $\exists \xi^* : \quad \|\xi^*\| \leq M \quad \text{and} \quad P(\xi^*) = 0$.

Our proofs will be based on the L_p regularity of elliptic problems with Neumann and slip boundary conditions. These are classical results, in case of cylindrical domain we only have to deal with the singularities of the boundary, for the sake of completeness we give the proofs. The first result concerns the Neumann problem.

Lemma 6. Let $1 < p < \infty$, $f \in L_p(\Omega)$, $d \in W_p^{1-1/p}(\Gamma)$ and $\int_{\Omega} f dx = \int_{\Gamma} d dS$. Let $C_0 \in \mathbb{R}$. Then there exists a solution $u \in W_p^2(\Omega)$ to the problem

$$\begin{aligned} \Delta u &= f & \text{in} & \quad \Omega, \\ \frac{\partial u}{\partial n} &= d & \text{on} & \quad \Gamma, \\ \frac{1}{|\Omega|} \int_{\Omega} u dx &= C_0, \end{aligned} \quad (2.6)$$

and

$$\|u - C_0\|_{W_p^2} \leq C [\|f\|_{L_p} + \|d\|_{W_p^{1-1/p}(\Gamma)}]. \quad (2.7)$$

Moreover, this solution is unique in the given regularity class.

Proof. In fact, the only difficulty we have to overcome are the singularities of the boundary. If $d|_{\Gamma_{in} \cup \Gamma_{out}} = 0$, then we can simply extend the weak solution using symmetric reflection that preserves the homogeneous Neumann condition. In the neighbourhood of Γ_{in} it will be defined as

$$E_s^v(u)(x) = \begin{cases} u(x), & x_1 \geq 0, \\ u(\tilde{x}), & x_1 < 0, \end{cases} \quad (2.8)$$

where $x = (x_1, x_2, x_3)$ and $\tilde{x} = (-x_1, x_2, x_3)$.

For $E_s^v(u)$ we obtain a Neumann problem in a domain with smooth boundary that we solve using classical elliptic theory.

If d does not vanish on $\Gamma_{in} \cup \Gamma_{out}$, we reduce the problem to the previous case. Let us consider Γ_{in} , on Γ_{out} we proceed the same way. Introducing an extension $u_0 \in W_p^2(Q)$ such that $\frac{\partial u_0}{\partial n}|_{\Gamma_{in}} = d$ and $\|u_0\|_{W_p^2} \leq C\|d\|_{W_p^{1-1/p}(\Gamma)}$, we consider $\tilde{u} = u - u_0$ that satisfies

$$\begin{aligned} \Delta \tilde{u} &= \tilde{f}, \\ \frac{\partial \tilde{u}}{\partial n}|_{\Gamma} &= 0, \end{aligned}$$

where $\|\tilde{f}\|_{L_p} \leq C[\|f\|_{L_p} + \|d\|_{W_p^{1-1/p}(\Gamma)}]$. □

The second result concerns the Lamé system with slip boundary conditions:

$$\begin{aligned} -\mu \operatorname{div}(\nabla u + \nabla^T u - \operatorname{div} u \mathbf{I}) - \lambda \nabla \operatorname{div} u &= F & \text{in } \Omega, \\ 2\mu(\mathbf{D}(u)n) \cdot \tau_i + \alpha u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma. \end{aligned} \quad (2.9)$$

The following lemma gives existence of a solution and the maximal regularity elliptic estimate that we apply in our proofs.

Lemma 7. *Under the assumptions of Theorem 1 there exists a unique $u \in W_p^2(\Omega)$ solving (2.9) with*

$$\|u\|_{W_p^2} \leq C[\|F\|_{L_p} + \sum_{i=1}^2 \|B_i\|_{W_p^{1-1/p}(\Gamma)}]. \quad (2.10)$$

Proof. Under the assumptions on μ and λ (2.9) is elliptic so we easily get a weak solution. The only problem we encounter showing regularity of the weak solution are the singularities of the boundary on the junctions of Γ_0 with Γ_{in} and Γ_{out} . These singularities can again be dealt using the symmetry arguments. Hence we define the operator E_{as}^v extending a vector field defined for $\{x : x_1 \geq 0\}$ on the whole space as

$$E_{as}^v(u)(x) = \begin{cases} u(x), & x_1 \geq 0, \\ \tilde{u}(\tilde{x}), & x_1 < 0, \end{cases} \quad (2.11)$$

where \tilde{x} is as above and $\tilde{u}(\tilde{x}) = [-u^1(x), u^2(x), u^3(x)]$. Then we have

$$\begin{aligned} -\operatorname{div}(\nabla E_{as}^v(v) + \nabla^T E_{as}^v(v) - \frac{2}{3} \operatorname{div} v \mathbf{I}) + \lambda \nabla \operatorname{div} E_{as}^v(v) \\ = -\operatorname{div}(\nabla v + \nabla^T v - \frac{2}{3} \operatorname{div} v \mathbf{I}) + \lambda \nabla \operatorname{div} v, \end{aligned} \quad (2.12)$$

and on the plane $x_1 = 0$ the extension E_{as}^v preserves the slip boundary conditions, i.e.

$$2\mu(\mathbf{D}(E_{as}^v(v))n) \cdot \tau_i + \alpha(E_{as}^v(v) \cdot \tau_i) = 2\mu(\mathbf{D}(v)n) \cdot \tau_i + \alpha(v \cdot \tau_i)$$

and $n \cdot E_{as}^v(v) = n \cdot v$. Now we can show higher regularity of $E_{as}^v(u)$, extension of the weak solution to (2.9), and since E_{as}^v preserves the boundary conditions we conclude that u is a strong solution to (2.9) satisfying the boundary conditions on Γ_{in} . Application of analogous antisymmetric extension on Γ_{out} completes the proof. \square

2.2 Remarks on the special solution

Our next aim is to construct solutions to (1.9). Recall that

$$\bar{\theta} = T_0 + (\bar{\theta}_0 - T_0) + \bar{\theta}_1,$$

where $\bar{\theta}_0$ solves (1.10). Therefore by Lemma 6 we have

$$\kappa \|\bar{\theta}_0 - T_0\|_{W_p^2} \leq C(c_g + \|g\|_{W_p^{1-1/p}(\Gamma)}) \leq C\|g\|_{W_p^{1-1/p}(\Gamma)}, \quad (2.13)$$

as well as by the Lax-Milgram theorem

$$\kappa \|\bar{\theta}_0 - T_0\|_{W_2^1} \leq C\|g\|_{L_2(\Gamma)}. \quad (2.14)$$

Next for $\bar{\theta}_1$ we have problem (1.13). Our aim is to show existence of a weak solution to this problem as well as its regularity. As the problem is linear, the existence and uniqueness of a weak solution is direct consequence of corresponding a priori estimates. First we show estimates in W_2^1 , then in W_p^2 . Since we will require κ and L large, we have to control how all constants depend on them. For the sake of simplicity we assume that κ and L are approximately of the same size.

Multiplying (1.13) by θ_1 and integrating over Ω we get

$$\begin{aligned} & \kappa \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx + \int_{\Gamma_0} L \bar{\theta}_1^2 dS + \frac{1}{2}(1 + e_2) \int_{\Gamma_{out}} \bar{\theta}_1^2 dS \\ &= \frac{1}{2}(1 + e_2) \int_{\Gamma_{in}} \bar{\theta}_1^2 dS + \int_{\Gamma_0} L \bar{\theta}_1 (T_1 + T_0 - \bar{\theta}_0) dS - \int_{\Omega} ((1 + e_2) \partial_{x_1} \bar{\theta}_0 + c_g) \bar{\theta}_1 dx. \end{aligned}$$

All terms on the l.h.s. are nonnegative. Applying the Poincaré inequality to the first term on the r.h.s. and the Young inequality to the other two ones we get

$$\begin{aligned} & \kappa \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx + L \int_{\Gamma_0} \bar{\theta}_1^2 dS \\ & \leq C_P \left(\int_{\Gamma_0} \bar{\theta}_1^2 dS + \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx \right) + \frac{L}{2} \int_{\Gamma_0} \bar{\theta}_1^2 dS + \frac{\kappa}{2} \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx \\ & \quad + C \left(L \int_{\Gamma_0} |T_1 + T_0 - \bar{\theta}_0|^2 dS + \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx + c_g^2 \right). \end{aligned}$$

Therefore we get, for κ and L sufficiently large with respect to C_p ,

$$\frac{\kappa}{4} \int_{\Omega} |\nabla \bar{\theta}_1|^2 dx + \frac{L}{4} \int_{\Gamma_0} \bar{\theta}_1^2 dS \leq C(\|g\|_{L_2(\Gamma)}^2 + L\|\bar{\theta}_0 - T_0\|_{W_2^1}^2 + L\|T_1\|_{L_2(\Gamma)}^2 + \|\bar{\theta}_0 - T_0\|_{L_2(\Gamma)}^2).$$

Hence, using also (2.14)

$$\|\bar{\theta}_1\|_{W_2^1} \leq C(\|g\|_{L_2(\Gamma)} + \|T_1\|_{L_2(\Gamma)}), \quad (2.15)$$

where C is a fixed constant independent of κ and L , provided $\kappa \sim L$. Uniqueness is a direct consequence of (2.15); if the data are zero, then the solution is zero as well.

Finally, proceeding similarly as above we will deduce the existence of strong solutions together with the estimates of the corresponding norms. First we rewrite (1.13) to the form

$$\begin{aligned} \kappa \Delta \bar{\theta}_1 &= (1 + e_2) \partial_{x_1} \bar{\theta}_1 + (1 + e_2) \partial_{x_1} \bar{\theta}_0 + c_g & \text{in } \Omega, \\ \kappa \frac{\partial \bar{\theta}_1}{\partial n} &= -L(\bar{\theta}_1 + \bar{\theta}_0 - T_0 - T_1) & \text{on } \Gamma, \end{aligned} \quad (2.16)$$

and using Lemma 6

$$\begin{aligned} \kappa \|\bar{\theta}_1\|_{W_p^2} &\leq C(\|\bar{\theta}_1\|_{W_p^1} + \|\bar{\theta}_0\|_{W_p^1} + \|g\|_{L_2(\Gamma)} \\ &+ L\|\bar{\theta}_1\|_{W_p^{1-1/p}(\Gamma)} + L\|T_1\|_{W_p^{1-1/p}(\Gamma)} + L\|\bar{\theta}_0 - T_0\|_{W_p^{1-1/p}(\Gamma)}). \end{aligned}$$

Now, applying the Poincaré inequality in the form

$$\|\bar{\theta}_1\|_{W_p^1} \leq \varepsilon \|\bar{\theta}_1\|_{W_p^2} + C(\varepsilon) \|\bar{\theta}_1\|_{W_2^1}$$

and using (2.15) we end up with

Lemma 8. *Let $T_1 \in W_p^{1-1/p}(\Gamma)$, $g \in W_p^{1-1/p}(\Gamma)$, $p > 3$ and let κ , L be sufficiently large. Then there exist a unique weak solution θ_1 to (1.13) such that*

$$\|\bar{\theta}_1\|_{W_2^1} \leq C[\|T_1\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}]. \quad (2.17)$$

Moreover, the solution is strong, i.e. $\bar{\theta}_1 \in W_p^2$ and

$$\|\bar{\theta}_1\|_{W_2^1} \leq C[\|T_1\|_{W_p^{1-1/p}(\Gamma)} + \|g\|_{W_p^{1-1/p}(\Gamma)}]. \quad (2.18)$$

Here, the constants C are independent of κ and L .

2.3 System for perturbations

As we are interested in small perturbations of $(\bar{v}, \bar{\rho}, \bar{\theta})$, it is natural to introduce the perturbations as unknown functions. For technical reasons it is better to get rid of inhomogeneity on the boundary. Hence we construct $u_0 \in W_p^2$ such that $u_0 \cdot n|_\Gamma = d$. More precisely, we set $u_0 = \nabla \phi$, where ϕ solves

$$\begin{aligned} \Delta \phi &= \int_\Gamma d \, dS & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} &= d & \text{on } \Gamma. \end{aligned}$$

Next we introduce the perturbations

$$u = v - \bar{v} - u_0, \quad \sigma = \rho - \bar{\rho}, \quad \eta = \theta - \bar{\theta} = \theta - \bar{\theta}_0 - \bar{\theta}_1. \quad (2.19)$$

Replacing the total energy balance (1.1)₃ with the internal energy balance (1.6) we can rewrite (1.1) as

$$\begin{aligned}
\partial_{x_1} u - \operatorname{div} \mathbf{S}(\nabla u) + p_1 \nabla \sigma + p_2 \nabla \eta &= F(u, \sigma, \eta) & \text{in } \Omega, \\
\operatorname{div} u + \sigma_{x_1} + (u + u_0) \cdot \nabla \sigma &= G(u, \sigma) & \text{in } \Omega, \\
(1 + e_2) \partial_{x_1} \eta - \kappa \Delta \eta + T_0 p_2 \operatorname{div} u &= H(u, \sigma, \eta) & \text{in } \Omega, \\
(\mathbf{S}(\nabla u) n) \cdot \tau_i + \alpha u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\
n \cdot u &= 0 & \text{on } \Gamma, \\
\sigma &= \sigma_{in} & \text{on } \Gamma_{in}, \\
\kappa \frac{\partial \eta}{\partial n} + L \eta &= 0 & \text{on } \Gamma,
\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
F(u, \sigma, \eta) &= (1 + \sigma) f - p_2 \nabla(\bar{\theta}_0 + \bar{\theta}_1) - (\partial_\rho \pi(\sigma + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta) - p_1) \nabla \sigma \\
&\quad (\partial_\theta \pi(\sigma + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta) - p_2) \nabla(\eta + \bar{\theta}_0 + \bar{\theta}_1) + \operatorname{div} \mathbf{S}(\nabla u_0) - \partial_{x_1} u_0 \\
&\quad -(u + u_0) \cdot \nabla(u + u_0) - \sigma(\bar{v} + u + u_0) \cdot \nabla(u + u_0), \\
G(u, \sigma) &= -(\sigma + 1) \operatorname{div} u_0 - \sigma \operatorname{div} u, \\
H(u, \sigma, \eta) &= \mathbf{S}(\nabla(u + u_0)) \cdot \nabla(u + u_0) - T_0 \partial_\rho \pi(1 + \sigma, \eta + \bar{\theta}_0 + \bar{\theta}_1) \operatorname{div} u_0 \\
&\quad -(1 + \sigma)(u + u_0) \cdot \nabla(\bar{\theta}_0 + \bar{\theta}_1 + \eta) + T_0 \operatorname{div} u (p_2 - \partial_\theta \pi(1 + \sigma, \bar{\theta}_0 + \bar{\theta}_1 + \eta)) \\
&\quad -(\eta + (\bar{\theta}_0 - T_0) + \bar{\theta}_1) \partial_\theta \pi(1 + \sigma, \bar{\theta}_0 + \bar{\theta}_1 + \eta) \operatorname{div}(u + u_0) \\
&\quad + \partial_{x_1}(\bar{\theta}_0 + \bar{\theta}_1 + \eta)(e_2 - \partial_\theta e(1 + \sigma, \bar{\theta}_0 + \bar{\theta}_1 + \eta)), \\
B_i &= b_i - \alpha \tau_i^{(1)} - (\mathbf{S}(\nabla u_0) n) \cdot \tau_i - \alpha u_0 \cdot \tau_i.
\end{aligned} \tag{2.21}$$

The above expressions result directly from substitution of (2.19) to (1.1). It is only worth to notice that to derive the expression for H we have applied the continuity equation which yields

$$\operatorname{div}(\rho e v) = \rho v \cdot \nabla e = \rho v \cdot \nabla \rho \partial_\rho e + \rho v \cdot \nabla \theta \partial_\theta e. \tag{2.22}$$

Now in (1.1)₃ the first term on the r.h.s. of (2.22) can be combined with the $\pi \operatorname{div} v$, what finally gives the above form of $H(\cdot)$.

Let us state at this early stage the following estimates on the quantities F, G, H defined in (2.21):

$$\begin{aligned}
\|F(u, \sigma, \eta)\|_{L_p} &\leq C[(\|u\|_{W_p^2} + \|\sigma\|_{W_p^1} + \|\eta\|_{W_p^2})^2 + (\|u\|_{W_p^2} + \|\sigma\|_{W_p^1})^3 + D_0(1 + D_0)], \\
\|G(u, \sigma)\|_{W_p^1} &\leq C[(\|u\|_{W_p^2} + \|\sigma\|_{W_p^1})^2 + D_0(1 + D_0)], \\
\|H(u, \sigma, \eta)\|_{L_p} &\leq C[(\|u\|_{W_p^2} + \|\sigma\|_{W_p^1} + \|\eta\|_{W_p^2})^2 \\
&\quad + (\|u\|_{W_p^2} + \|\sigma\|_{W_p^1} + \|\eta\|_{W_p^2})^3 + D_0(1 + D_0)],
\end{aligned} \tag{2.23}$$

where D_0 is defined in (1.14). The bounds (2.23) are straightforward, it is enough to observe that F, G and H are composed of terms which are either quadratic or cubic with respect to the functions u, σ, η or linear with respect to them multiplied by something which can be bounded

by D_0 . Note that in the estimate of terms contained differences of the derivatives of π or e we use the fact that these functions are twice continuously differentiable and hence we may use the Taylor expansion theorem.

We sketch briefly the idea of the proof of Theorem 1, and hence the structure of the remainder of the paper. In Subsection 2.4 we explain how to deal with the linearization of the continuity equation, i.e. equation (2.20)₂. To show existence of solutions we can apply a method of successive approximations and look for the solution as a limit of a sequence

$$\begin{aligned}
& \partial_{x_1} u^{n+1} - \mathbf{S}(\nabla u^{n+1}) + p_1 \nabla \sigma^{n+1} + p_2 \nabla \eta^{n+1} = F(u^n, \sigma^n, \eta^n) & \text{in } \Omega, \\
& \operatorname{div} u^{n+1} + \sigma_{x_1}^{n+1} + (u^n + u_0) \cdot \nabla \sigma = G(u^n, \sigma^n) & \text{in } \Omega, \\
& (1 + e_2) \partial_{x_1} \eta^{n+1} - \kappa \Delta \eta^{n+1} + T_0 p_2 \operatorname{div} u^{n+1} = H(u^n, \sigma^n, \eta^n) & \text{in } \Omega, \\
& (\mathbf{S}(\nabla u^{n+1})n) \cdot \tau_i + \alpha u^{n+1} \cdot \tau_i = B_i, \quad i = 1, 2 & \text{on } \Gamma, \\
& n \cdot u^{n+1} = 0 & \text{on } \Gamma, \\
& \sigma^{n+1} = \sigma_{in} & \text{on } \Gamma_{in}, \\
& \kappa \frac{\partial \eta^{n+1}}{\partial n} + L \eta^{n+1} = 0 & \text{on } \Gamma.
\end{aligned} \tag{2.24}$$

In Section 3 we deal with the linear system (3.1) showing the a priori estimates. The first step is the energy estimate which is derived in quite a standard way. The energy estimate is then applied to show the maximal regularity L^p estimates, hence estimates in the space where we look for the solution (Lemma 12). It can be considered as the main result of Section 3. In the end of Section 3 we solve the linear system, hence we show that the sequence (2.24) is well defined. In Section 4 we deal with the sequence of approximations. Using the estimates for the linear system combined with (2.21) we show it is bounded in $W_p^2 \times W_p^1 \times W_p^2$ and has a contraction property in a weaker space $W_2^1 \times L_\infty(L_2) \times W_2^1$. An analogous estimate gives uniqueness in the class of solutions satisfying (1.15). In other words we apply a generalization of the Banach fixed point theorem to show the convergence of the sequence of approximations to the solution of (1.1).

2.4 Steady transport equation

In this section we show the well posedness of the operator $S : L_2 \rightarrow L_\infty(L_2)$ defined as

$$w = S(h) \iff \begin{cases} \partial_{x_1} w + U \cdot \nabla w = h & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in}, \end{cases} \tag{2.25}$$

where w is a weak solution to (2.25) in the sense as introduced in (4.3). The solvability of the above problem, to which we refer as a steady transport equation, will be needed to show the energy estimate for the density in Section 3 and to solve the continuity equation in Section 4. The result is given in the following

Lemma 9. *Let $h \in L_2(\Omega)$ and $w_{in} \in L_2(\Gamma_{in})$. Let $U \in W_p^2(\Omega)$ be such that $\|U\|_{W_p^2}$ is small enough and $U \cdot n|_{\Gamma_0} = 0$. Then there exists a unique solution $w = S(h) \in L_\infty(L_2)$ and*

$$\|S(h)\|_{L_\infty(L_2)} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})} + \|h\|_{L_2(\Omega)}]. \tag{2.26}$$

The proof follows [23]. For the sake of completeness we recall here the most important steps, referring to [23] for the missing details. The idea is to get rid of the term $U \cdot \nabla \sigma$ introducing a change of variables $x = \psi(z)$ satisfying the identity

$$\partial_{z_1} = \partial_{x_1} + U \cdot \nabla_x. \quad (2.27)$$

We construct the mapping ψ in the following

Lemma 10. *Let $\|U\|_{W_p^2}$ be small enough. Then there exists a set $\mathcal{O} \subset \mathbb{R}^3$ and a diffeomorphism $x = \psi(z)$ defined on \mathcal{O} such that $\Omega = \psi(\mathcal{O})$ and (2.27) holds. Moreover, if $z_n \rightarrow z$ and $\psi(z_n) \rightarrow \Gamma_0$ then $n^1(z) = 0$, where n is the outward normal to \mathcal{O} .*

Before we sketch the proof let us make one remark. The last condition states that the first component of the normal to $\psi^{-1}(\Gamma_0)$ vanishes, but since ψ is defined only on \mathcal{O} we formulate this condition using the limits. It means simply that the image $\mathcal{O} = \psi^{-1}(\Omega)$ is also a cylinder with a flat wall.

Sketch of the proof of Lemma 10. The identity (2.27) means that ψ must satisfy

$$\frac{\partial \psi^1}{\partial z_1} = 1 + U^1(\psi), \quad \frac{\partial \psi^2}{\partial z_1} = U^2(\psi), \quad \frac{\partial \psi^3}{\partial z_1} = U^3(\psi). \quad (2.28)$$

A natural condition is that $\psi(\Gamma_{in}) = \Gamma_{in}$. Thus we can search for $\psi(z_1, z_2, z_3) = \psi_{z_2, z_3}(z_1)$, where for all (z_2, z_3) such that $(z_2, z_3, 0) \in \Gamma_{in}$ the function $\psi_{z_2, z_3}(\cdot)$ is a solution to a system of ODE:

$$\begin{cases} \partial_s \psi_{z_2, z_3}^1 = 1 + U^1(\psi_{z_2, z_3}), & \partial_s \psi_{z_2, z_3}^2 = U^2(\psi_{z_2, z_3}), & \partial_s \psi_{z_2, z_3}^3 = U^3(\psi_{z_2, z_3}), \\ \psi_{z_2, z_3}(0) = (0, z_2, z_3). \end{cases} \quad (2.29)$$

Using the regularity and smallness of U we show existence for (2.29), hence we have ψ defined on some \mathcal{O} such that $\Omega = \psi(\mathcal{O})$. The smallness of U implies also that $\psi(z) = z + \psi_\epsilon(z)$, where $\|\psi_\epsilon\|_{W_\infty^1}$ is small and we conclude that ψ is a diffeomorphism. Let us denote $\phi = \psi^{-1}$. Now it is natural to define the subsets of $\partial\mathcal{O}$ as $\partial\mathcal{O} = \mathcal{O}_{in} \cup \mathcal{O}_{out} \cup \mathcal{O}_0$ where $\mathcal{O}_{in} = \Gamma_{in}$, $\mathcal{O}_{out} = \{z : z = \lim \phi(x_n), x_n \rightarrow \Gamma_{out}\}$ and $\mathcal{O}_0 = \{z : z = \lim \phi(x_n), x_n \rightarrow \Gamma_0\}$. To show that $n^1(z) = 0$ for $z \in \mathcal{O}_0$ it is enough to observe that

$$D\psi(z)([1, 0, 0]) = [1 + U^1(x), U^2(x), U^3(x)],$$

where $x = \psi(z)$. But for $x \in \Gamma_0$ the vector on the r.h.s is tangent to Γ_0 since $U \cdot n|_{\Gamma_0} = 0$. We can conclude that on \mathcal{O}_0 the image in ψ of a straight line $\{(s, z_2, z_3) : s \in (0, b)\}$ is a curve tangent to Γ_0 , and thus \mathcal{O}_0 is a sum of such lines and so we have $n^1(z) = 0$. The proof of lemma 10 is completed. \square

Proof of Lemma 9. First we define $S(h)$ for a continuous function h as

$$S(h)(x) = w_{in}(0, \phi_2(x), \phi_3(x)) + \int_0^{\phi_1(x)} h(\psi(s, \phi_2(x), \phi_3(x))) ds. \quad (2.30)$$

The condition $n^1 = 0$ on $\phi(\Gamma_0)$ guarantees that a straight line $(s, z_1, z_2) : s \in (0, b)$ has a picture in Ω and thus we integrate along a curve contained in Ω . It means that S is well defined for continuous functions defined on Ω and the construction of ψ clearly ensures that S satisfies (2.25). Next we have to extend S on $L_2(\Omega)$. To this end we show the estimate (2.26) for continuous h . Let Ω_{x_1} denote an x_1 -cut of Ω and let $\bar{x} := (x_2, x_3)$. Then by (2.30) we have

$$\begin{aligned} \|S(h)\|_{L_2(\Omega_{x_1})}^2 &= \int_{\Omega_{x_1}} \left[w_{in}(0, \phi_2(x), \phi_3(x)) + \int_0^{\phi_1(x)} h(\psi(s, \phi_2(x), \phi_3(x))) ds \right]^2 d\bar{x} \\ &\leq 2\|w_{in}\|_{L_2(\Gamma_{in})}^2 + C \int_{\Omega_{x_1}} \int_0^{\phi_1(x)} h^2(\psi(s, \phi_2(x), \phi_3(x))) ds d\bar{x} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})}^2 + \|h\|_{L_2(\Omega)}^2]. \end{aligned}$$

The above estimate holds for every $x_1 \in (0, L)$ what implies (2.26). Now we can define $S(v)$ for $v \in L_2(\Omega)$ using a standard density argument, the details are given in ([23]). To show the uniqueness of the solution we can rewrite the r.h.s of (2.25) as

$$\begin{cases} \partial_{x_1} w + \frac{U^2}{1+U^1} \partial_{x_2} w + \frac{U^3}{1+U^1} \partial_{x_3} w = \frac{h}{1+U^1} & \text{in } \Omega, \\ w = w_{in} & \text{on } \Gamma_{in}, \end{cases} \quad (2.31)$$

where the equation is to be understood in the sense of distribution, and, treating x_1 as a "time" variable, adapt Di Perna - Lions theory of transport equation ([5]) that implies the uniqueness of solution to (2.31) in the class $L_\infty(L_2)$. The proof is thus complete. \square

3 Linearization and a priori bounds

In this section we deal with the linear system

$$\begin{aligned} \partial_{x_1} u - \operatorname{div} \mathbf{S}(\nabla u) + p_1 \nabla \sigma + p_2 \nabla \eta &= F & \text{in } \Omega, \\ \operatorname{div} u + \partial_{x_1} \sigma + U \cdot \nabla \sigma &= G & \text{in } \Omega, \\ r_0 \partial_{x_1} \eta + r_1 \operatorname{div} u - \kappa \Delta \eta &= H, & \text{in } \Omega, \\ (\mathbf{S}(\nabla u) n) \cdot \tau_i + \alpha u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma, \\ \sigma &= \sigma_{in} & \text{on } \Gamma_{in}, \\ \kappa \frac{\partial \eta}{\partial n} + L \eta &= 0 & \text{on } \Gamma \end{aligned} \quad (3.1)$$

with given functions $(F, G, H, B_i, \sigma_{in}, U) \in (L_p \times W_p^1 \times L_p \times W_p^{1-1/p}(\Gamma) \times W_p^1(\Gamma_{in}) \times W_p^2)$ such that

$$U \cdot n = 0 \text{ on } \Gamma_0, \quad U \cdot n > 0 \text{ on } \Gamma_{out}, \quad U \cdot n < 0 \text{ on } \Gamma_{in}. \quad (3.2)$$

Note that this system actually represents (2.20), where we only redefined a few constants. The main difficulty lies in deriving appropriate estimates for (3.1). We start with the energy estimates, then applying the properties of slip boundary conditions, Helmholtz decomposition of the velocity and classical elliptic theory we derive the L_p estimates.

Apart from solving the system (3.1), the estimates derived in this section are used later, combined with (2.23), to show the convergence of the sequence of approximations (2.24).

3.1 Energy estimates

Lemma 11. *Let (u, σ, η) solve system (3.1) with $(F, G, H, B_i, \sigma_{in}, U) \in V^* \times L_2 \times L_2 \times L_2(\Gamma) \times L_2(\Gamma_{in}) \times W_p^2$ with $\|U\|_{W_p^2}$ small enough, $U \cdot n$ on Γ fulfilling conditions (3.2) above, where V^* is the dual space to*

$$V = \{v \in W_2^1(\Omega) : v \cdot n|_\Gamma = 0\}. \quad (3.3)$$

Assume further that κ, L and α fulfill the assumptions of Theorem 1. Then the following estimate holds:

$$\begin{aligned} & \|u\|_{W_2^1} + \|\sigma\|_{L_\infty(L_2)} + \|\eta\|_{W_2^1} \\ & \leq C[\|F\|_{V^*} + \|G\|_{L_2} + \|H\|_{L_2} + \|B_i\|_{L_2(\Gamma)} + \|\sigma_{in}\|_{L_2(\Gamma_{in})}]. \end{aligned} \quad (3.4)$$

Proof. We multiply (3.1)₁ by u and integrate applying the following identity:

$$\int_\Omega -\operatorname{div} \mathbf{S}(u) v \, dx = \int_\Omega \mathbf{S}(u) : \nabla v \, dx - \int_\Gamma (\mathbf{S}(u)n) \cdot v \, dS. \quad (3.5)$$

With application of the Korn inequality and the boundary condition (3.1)₄ we get

$$\begin{aligned} & C\|u\|_{W_1^2}^2 - \int_\Omega \eta \operatorname{div} u \, dx \leq \int_\Omega F \cdot u \, dx + \int_\Omega G \sigma \, dx \\ & + \frac{p_1}{2} \int_\Omega \sigma^2 \operatorname{div} U \, dx + \frac{p_1}{2} \int_{\Gamma_{in}} \sigma_{in}^2 (1 - U \cdot n) \, dS + \sum_{i=1}^2 \int_\Gamma B_i u \cdot \tau_i \, dS. \end{aligned} \quad (3.6)$$

Next we divide (3.1)₃ by r_1 (recall it is constant), multiply by η and integrate. It yields

$$\begin{aligned} & \frac{\kappa}{r_1} \int_\Omega |\nabla \eta|^2 \, dx + \frac{L}{r_1} \int_{\Gamma_0} \eta^2 \, dS \\ & - \frac{r_0}{2r_1} \int_{\Gamma_{in}} \eta^2 \, dS + \frac{r_0}{2r_1} \int_{\Gamma_{out}} \eta^2 \, dS + \int_\Omega \eta \operatorname{div} u \, dx = \int_\Omega H \eta \, dx, \end{aligned}$$

hence

$$\frac{\kappa}{r_1} \int_\Omega |\nabla \eta|^2 \, dx + \frac{L}{r_1} \int_{\Gamma_0} \eta^2 \, dS + \int_\Omega \eta \operatorname{div} u \, dx \leq \int_\Omega H \eta \, dx + \frac{r_0}{2r_1} \int_{\Gamma_{in}} \eta^2 \, dS. \quad (3.7)$$

To get rid of the boundary term on the r.h.s we can apply the Poincaré inequality (2.1) and rewrite (3.7) as

$$\frac{1}{r_1} \left(\kappa - \frac{C_P}{2} \right) \int_\Omega |\nabla \eta|^2 \, dx + \frac{1}{r_1} \left(L - \frac{C_P}{2} \right) \int_{\Gamma_0} \eta^2 \, dS + \int_\Omega \eta \operatorname{div} u \, dx \leq \int_\Omega H \eta \, dx, \quad (3.8)$$

where C_P is the constant from (2.1). For κ large enough and L large enough on Γ_0 the first two terms on the l.h.s. will be positive. Now we can combine (3.6) and (3.7) obtaining

$$\begin{aligned} & C(\Omega, \kappa, L) [\|u\|_{W_2^1}^2 + \int_\Omega |\nabla \eta|^2 \, dx + \|\eta\|_{L_2(\Gamma_0)}^2] \leq \int_\Omega H \eta \, dx + \int_\Omega F \cdot u \, dx \\ & + \int_\Omega G \sigma \, dx + \frac{1}{2} \int_\Omega \sigma^2 \operatorname{div} U \, dx + \frac{1}{2} \int_{\Gamma_{in}} \sigma_{in}^2 (1 - U \cdot n) \, dS + \sum_{i=1}^2 \int_\Gamma B_i u \cdot \tau_i \, dS. \end{aligned}$$

Using (2.2), the imbedding $W_p^1 \hookrightarrow L_\infty$ and Hölder's inequality we derive

$$\begin{aligned} & \|u\|_{W_2^1}^2 + \|\eta\|_{W_2^1}^2 \leq \|F\|_{V^*} \|u\|_{W_2^1} + \|G\|_{L_2} \|\sigma\|_{L_2} \\ & + \|H\|_{L_2} \|\eta\|_{L_2} + C \|\sigma_{in}\|_{L_2(\Gamma_{in})}^2 + E \|\sigma\|_{L_2}^2 + \sum_{i=1}^2 \|B_i\|_{L_2(\Gamma)} \|u\|_{L_2(\Gamma)}. \end{aligned} \quad (3.9)$$

Recall that E denotes a small constant and so to complete the proof we have to find the bound on $\|\sigma\|_{L_\infty(L_2)}$. But this follows from the existence result in the previous subsection, see (2.26). Hence we have

$$\|\sigma\|_{L_\infty(L_2)} \leq C \left(\|u\|_{W_2^1} + \|G\|_{L_2} + \|\sigma_{in}\|_{L_2(\Gamma_{in})} \right). \quad (3.10)$$

Combining (3.9) and (3.10) we conclude (3.4). \square

3.2 L_p estimates

The main result of this section is

Lemma 12. *Let (u, σ, η) solve system (3.1) with $(F, G, H, B_i, \sigma_{in}, U) \in L_p \times W_p^1 \times L_p \times W_p^{1-1/p}(\Gamma) \times W_p^1(\Gamma_{in}) \times W_p^2$, where $\|U\|_{W_p^2}$ is small enough and κ, L, α satisfy the assumptions of Theorem 1. Then*

$$\begin{aligned} & \|u\|_{W_p^2} + \|\sigma\|_{W_p^1} + \|\eta\|_{W_p^2} \\ & \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|H\|_{L_p} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})} + \sum_{i=1}^2 \|B_i\|_{W^{1-1/p}(\Gamma)}] := D_{lin}. \end{aligned} \quad (3.11)$$

The proof of (3.11) will be performed in several consecutive lemmas. We show the bound on $\|\eta\|_{W_p^2}$. Then we proceed with the bound on $\|\sigma\|_{W_p^1}$ which is the most demanding. First we show bound on the vorticity which, together with Helmholtz decomposition of the velocity, makes possible to eliminate the term $\operatorname{div} u$ from the continuity equation, leading to (3.23). Using this equation we show the bound on the density. Then (3.11) easily follows from the classical elliptic estimate on $\|u\|_{W_p^2}$.

We start with the bound on η ; to show it let us rewrite (3.1)_{3,7} as

$$\begin{aligned} -\kappa \Delta \eta &= H - r_0 \partial_{x_1} \eta - r_1 \operatorname{div} u & \text{in } \Omega, \\ \frac{\partial \eta}{\partial n} \Big|_{\Gamma} &= -\frac{L}{\kappa} \eta & \text{on } \Gamma. \end{aligned} \quad (3.12)$$

The classical elliptic estimate ([2], [3]) for the above system yields

$$\|\eta\|_{W_p^2} \leq C [\|H\|_{L_p} + \|\eta\|_{W_p^1} + \|\eta\|_{W_p^{1-1/p}(\Gamma)} + \|u\|_{W_p^1}].$$

Applying the trace theorem to the boundary term and then the interpolation inequality (2.3) we get

$$\|\eta\|_{W_p^2} \leq C [\|H\|_{L_p} + \|\eta\|_{W_2^1} + \|u\|_{W_2^1}] + \epsilon \|u\|_{W_p^2}. \quad (3.13)$$

By the energy estimate (3.4) we conclude

$$\|\eta\|_{W_p^2} \leq C D_{lin} + \epsilon \|u\|_{W_p^2}. \quad (3.14)$$

Now we proceed towards the bound on $\|\sigma\|_{W_p^1}$. As mentioned before, we start with the estimate on the vorticity. Taking the curl of (3.1)₁ we get

$$\begin{aligned} -\mu \Delta \alpha &= \operatorname{curl} [F - \partial_{x_1} u] & \text{in } \Omega, \\ \alpha \cdot \tau_2 &= (2\chi_1 - \frac{\alpha}{\mu}) u \cdot \tau_1 + \frac{B_1}{\mu} & \text{on } \Gamma, \\ \alpha \cdot \tau_1 &= (\frac{\alpha}{\mu} - 2\chi_2) u \cdot \tau_2 - \frac{B_2}{\mu} & \text{on } \Gamma, \\ \operatorname{div} \alpha &= 0 & \text{on } \Gamma, \end{aligned} \quad (3.15)$$

where χ_i denote the curvatures of the curves generated by tangent vectors τ_i . In order to show the boundary relations (3.15)_{2,3} it is enough to differentiate (3.1)₄ with respect to the tangential directions and apply (3.1)₃. A rigorous proof is given in [19] or [23].

The condition $\operatorname{div} \alpha = 0$ in Ω results simply from the fact that $\alpha = \operatorname{curl} u$. We introduce this relation as a boundary condition $(3.15)_4$, that completes the conditions on the tangential parts of the vorticity. Note that the boundary conditions $(3.15)_{2,3}$ give the tangential parts of the vorticity on the boundary the regularity of the velocity itself and the data. We will now use this feature of slip boundary conditions to show the higher estimate on the vorticity (see [14],[15], [26]).

For the above system we have (see [30], Theorem 10.4):

$$\|\alpha\|_{W_p^1} \leq C \left[\|F\|_{L_p} + \|\partial_{x_1} u\|_{L_p} + \sum_{i=1}^2 \|B_i\|_{W_p^{1-1/p}(\Gamma)} + \|u\|_{W_p^{1-1/p}(\Gamma)} \right]. \quad (3.16)$$

Applying the interpolation inequality (2.3) and then (3.4) we arrive at

$$\|\alpha\|_{W_p^1} \leq C(\epsilon) D_{lin} + \epsilon \|u\|_{W_p^2} \quad (3.17)$$

for any $\epsilon > 0$, where D_{lin} is defined in (3.11). Now we introduce the Helmholtz decomposition of the velocity (see e.g. [7])

$$u = \nabla \phi + A, \quad (3.18)$$

where $\frac{\partial \phi}{\partial n}|_{\Gamma} = 0$ and $\operatorname{div} A = 0$. We see that the field A satisfies the following system

$$\begin{aligned} \operatorname{curl} A &= \alpha & \text{in } \Omega, \\ \operatorname{div} A &= 0 & \text{in } \Omega, \\ A \cdot n &= 0 & \text{on } \Gamma. \end{aligned} \quad (3.19)$$

For this system we have (see [27]): $\|A\|_{W_p^2} \leq C \|\alpha\|_{W_p^1}$, what by (3.16) can be rewritten as

$$\|A\|_{W_p^2} \leq C(\epsilon) \left[\|F\|_{L_p} + \sum_{i=1}^2 \|B_i\|_{W_p^{1-1/p}(\Gamma)} \right] + \epsilon \|u\|_{W_p^2} \quad (3.20)$$

for any $\epsilon > 0$. Now we substitute the Helmholtz decomposition to $(3.1)_1$. We get

$$\nabla [-(\lambda + \frac{4}{3}\mu)\Delta \phi + p_1 \sigma] = F - \partial_{x_1} A + \mu \Delta A - \partial_{x_1} \nabla \phi - p_2 \nabla \eta, \quad (3.21)$$

but $\Delta \phi = \operatorname{div} u$. We denote

$$-(\lambda + \frac{4}{3}\mu)\operatorname{div} u + p_1 \sigma = \bar{K}. \quad (3.22)$$

Combining the last equation with $(3.1)_2$ we arrive at

$$\gamma \sigma + \partial_{x_1} \sigma + U \cdot \nabla \sigma = K, \quad (3.23)$$

where $\gamma = \frac{p_1}{\lambda + \frac{4}{3}\mu}$ and

$$K = \frac{\bar{K}}{\lambda + \frac{4}{3}\mu} + G. \quad (3.24)$$

Equation (3.23) makes possible to estimate the W_p^1 -norm of the density in terms of W_p^1 - norm of K , which in turn will be controlled by (3.21) using interpolation and the energy estimate (3.6). First we estimate $\|\sigma\|_{W_p^1}$ in terms of K . The result is stated in the following lemma

Lemma 13. Assume that σ satisfies equation (3.23) with $K \in W_p^1$. Then

$$\|\sigma\|_{W_p^1} \leq C [\|K\|_{W_p^1} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (3.25)$$

Proof. In order to find a bound on $\|\sigma\|_{L_p}$ we multiply (3.23) by $|\sigma|^{p-2}\sigma$ and integrate over Ω . Integrating by parts and next using the boundary conditions we get

$$\begin{aligned} \gamma \|\sigma\|_{L_p}^p - \frac{1}{p} \int_{\Omega} \operatorname{div} U |\sigma|^p dx + \frac{1}{p} \int_{\Gamma_{out}} |\sigma|^p (1 + U \cdot n) dS \\ \leq \|K\|_{L_p} \|\sigma\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |\sigma_{in}|^p (1 - U \cdot n) dS. \end{aligned} \quad (3.26)$$

With the smallness of $\|U\|_{W_p^2}$, the above implies

$$C \|\sigma\|_{L_p}^p \leq \|K\|_{L_p} \|\sigma\|_{L_p}^{p-1} + C \|\sigma_{in}\|_{L_p(\Gamma_{in})}^p,$$

and so

$$\|\sigma\|_{L_p} \leq C [\|K\|_{L_p} + \|\sigma_{in}\|_{L_p(\Gamma_{in})}]. \quad (3.27)$$

Now we estimate the derivatives. In order to find a bound on $\partial_{x_i}\sigma$ we differentiate (3.23) with respect to x_i . Note that $\nabla^2\sigma$ is not defined in general, however, for

$$\tilde{u} := [1 + U^1, U^2, U^3] \quad (3.28)$$

and $\sigma \in W_p^1$ we may set $\tilde{u} \cdot \nabla \partial_{x_i}\sigma := \partial_{x_i}(\tilde{u} \cdot \nabla \sigma) - \partial_{x_i}\tilde{u} \cdot \nabla \sigma \in L_p$.

Hence we can differentiate (3.23) with respect to x_i , multiply by $|\partial_{x_i}\sigma|^{p-2}\partial_{x_i}\sigma$ and integrate. Since $\partial_{x_i}\tilde{u} = \partial_{x_i}U$, we have

$$\left| \int_{\Omega} \partial_{x_i}\tilde{u} \cdot (|\partial_{x_i}\sigma|^{p-2}\partial_{x_i}\sigma \nabla \sigma) dx \right| \leq \|\nabla U\|_{L_{\infty}} \|\nabla \sigma\|_{L_p}^p \leq C \|U\|_{W_p^2} \|\nabla \sigma\|_{L_p}^p.$$

Next, since $\tilde{u} \cdot \nabla \partial_{x_i}\sigma \in L_p$, we can write

$$\begin{aligned} \int_{\Omega} \tilde{u} \cdot |\partial_{x_i}\sigma|^{p-2}\partial_{x_i}\sigma \nabla \partial_{x_i}\sigma dx &= \frac{1}{p} \int_{\Omega} \tilde{u} \cdot \nabla |\partial_{x_i}\sigma|^p dx \\ &= -\frac{1}{p} \int_{\Omega} |\partial_{x_i}\sigma|^p \operatorname{div} \tilde{u} dx + \frac{1}{p} \int_{\Gamma} |\partial_{x_i}\sigma|^p \tilde{u} \cdot n dS \\ &= -\frac{1}{p} \int_{\Omega} |\partial_{x_i}\sigma|^p \operatorname{div} U dx - \frac{1}{p} \int_{\Gamma_{in}} |\partial_{x_i}\sigma_{in}|^p (1 + U^1) dS + \frac{1}{p} \int_{\Gamma_{out}} |\partial_{x_i}\sigma|^p (1 + U^1) dS. \end{aligned}$$

For $i = 2, 3$ we have $\sigma_{in, x_i} \in L_p(\Gamma_{in})$ and hence the above defines the trace of $|\partial_{x_i}\sigma|^p$ on Γ_{out} . We arrive at

$$\begin{aligned} \gamma \|\partial_{x_i}\sigma\|_{L_p}^p - \frac{1}{p} \int_{\Omega} \operatorname{div} U |\partial_{x_i}\sigma|^p dx + \frac{1}{p} \int_{\Gamma_{out}} |\partial_{x_i}\sigma|^p (1 + U^1) dS \\ \leq \|\partial_{x_i}K\|_{L_p} \|\partial_{x_i}\sigma\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |\partial_{x_i}\sigma_{in}|^p (1 + U^1) dS + C \|U\|_{W_p^2} \|\nabla \sigma\|_{L_p}^p. \end{aligned} \quad (3.29)$$

For $i = 2, 3$ (3.29) gives straightforward bound on $\|\partial_{x_i}\sigma\|_{L_p}$. In order to estimate $\partial_{x_1}\sigma$ we also differentiate (3.23) with respect to x_1 and multiply by $|\partial_{x_1}\sigma|^{p-2}\partial_{x_1}\sigma$. The difference is that $\partial_{x_1}\sigma$ is not given on Γ_{in} . To overcome this difficulty we can observe that on Γ_{in} equation (3.23) reduces to

$$\gamma \sigma_{in} + U^2 \partial_{x_2}\sigma_{in} + U^3 \partial_{x_3}\sigma_{in} + [1 + U^1] \partial_{x_1}\sigma = K,$$

what can be rewritten as

$$\partial_{x_1}\sigma = \frac{1}{1+U^1} [K - \gamma\sigma_{in} - U_\tau \cdot \nabla_\tau \sigma_{in}],$$

where the lower index τ denotes the tangential component. Thus we have

$$\|\partial_{x_1}\sigma\|_{L_p(\Gamma_{in})} \leq C [\|K|_{\Gamma_{in}}\|_{L_p(\Gamma_{in})} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}].$$

Using this bound in (3.29), $i = 1$, we arrive at the estimate

$$\|\partial_{x_1}\sigma\|_{L_p}^p \leq C [\|\partial_{x_1}K\|_{L_p} \|\partial_{x_1}\sigma\|_{L_p}^{p-1} + \|U\|_{W_p^2} \|\nabla\sigma\|_{L_p}^p + \|K\|_{L_p(\Gamma_{in})}^p + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}^p]. \quad (3.30)$$

Applying the trace theorem to the term $\|K\|_{L_p(\Gamma_{in})}$ and then combining (3.29) (for x_2 and x_3) with (3.30) we get

$$\|\nabla\sigma\|_{L_p}^p \leq C [\|\nabla K\|_{L_p} \|\nabla\sigma\|_{L_p}^{p-1} + \|U\|_{W_p^2} \|\nabla\sigma\|_{L_p}^p + \|K\|_{W_p^1}^p + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}^p]. \quad (3.31)$$

The term $\|U\|_{W_p^2} \|\nabla\sigma\|_{L_p}^p$ can be put on the l.h.s. due to the smallness assumption and thus by Young's inequality

$$\|\nabla\sigma\|_{L_p} \leq C [\|K\|_{W_p^1} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}], \quad (3.32)$$

what combined with (3.27) yields

$$\|\sigma\|_{W_p^1} \leq C [\|K\|_{W_p^1} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (3.33)$$

The lemma is proved. \square

Now we estimate K in terms of the data. The result is

Lemma 14. *Let K be defined in (3.24). Then $\forall \delta > 0$ we have*

$$\|K\|_{W_p^1} \leq \delta \|u\|_{W_p^2} + C(\delta) D_{lin}, \quad (3.34)$$

where D_{lin} is defined in (3.11).

Proof. Applying first (2.3) and then (3.6) we get

$$\|K\|_{L_p} \leq \delta_1 \|\nabla K\|_{L_p} + C(\delta_1) [\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)}]. \quad (3.35)$$

Hence, it is enough to find the bound on $\|\nabla K\|_{L_p}$. By (3.21) we have

$$\|\nabla K\|_{L_p} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|A\|_{W_p^2} + \|\partial_{x_1}\nabla\phi\|_{L_p} + \|\nabla\eta\|_{L_p}],$$

where $u = \nabla\phi + A$ is the Helmholtz decomposition of the velocity. Applying (3.20), (3.14), trace theorem and (2.3) we arrive at

$$\begin{aligned} \|\nabla K\|_{L_p} &\leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|H\|_{L_p} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}] \\ &\quad + \delta_1 \|u\|_{W_p^2} + C(\delta_1) [\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)}]. \end{aligned} \quad (3.36)$$

Combining (3.35) and (3.36) we get (3.34). \square

The bound on σ now follows directly. Substituting (3.34) to (3.25) we get

$$\|\sigma\|_{W_p^1} \leq \delta \|u\|_{W_p^2} + C(\delta) D_{lin}, \quad (3.37)$$

where D_{lin} is given by (3.11). Now the only missing piece to complete the proof of (3.11) is the bound on $\|u\|_{W_p^2}$. To show it note that the velocity satisfies the Lamé system

$$\begin{aligned} u_{x_1} - \operatorname{div} \mathbf{S}(\nabla u) &= F - p_1 \nabla \sigma - p_2 \nabla \eta && \text{in } \Omega, \\ (\mathbf{S}(\nabla u)n) \cdot \tau_i + \alpha u \cdot \tau_i &= B_i, \quad i = 1, 2 && \text{on } \Gamma, \\ n \cdot u &= 0 && \text{on } \Gamma. \end{aligned} \quad (3.38)$$

The classical theory of elliptic equations ([2],[3]) yields

$$\|u\|_{W_p^2} \leq C [D_{lin} + \|u\|_{W_p^1}].$$

Applying (2.3) to the term $\|u\|_{W_p^1}$ and then (3.6) we get

$$\|u\|_{W_p^2} \leq C D_{lin}. \quad (3.39)$$

Proof of Lemma 12. We combine (3.14), (3.39) and (3.37). In (3.37) we choose for example $\delta = \frac{1}{2C}$ where C is the constant from (3.39). Now (3.11) follows immediately. \square

4 Solution of the linear system

In this section we solve the linear system (3.1) and hence show that the sequence (2.24) is well defined. We start with showing the existence of an appropriately defined weak solution applying the Galerkin method modified in a way to deal with the continuity equation. Then we show the regularity of the solutions for the given regularity of the data. For simplicity let us denote

$$\tilde{u} := [1 + U^1, U^2, U^3]. \quad (4.1)$$

4.1 Weak solution

In order to define a weak solution to (3.1) we recall the definition of the space V (3.3). A natural definition of a weak solution to the system (3.1) is $(u, \sigma, \eta) \in V \times L_\infty(L_2) \times W_2^1$ such that

1. the identity

$$\begin{aligned} \int_{\Omega} \{v \cdot \partial_{x_1} u + \mathbf{S}(\nabla u) : \nabla v - p_1 \sigma \operatorname{div} v - p_2 \eta \operatorname{div} v\} dx + \sum_{i=1}^2 \int_{\Gamma} \alpha (u \cdot \tau_i) (v \cdot \tau_i) dS \\ = \int_{\Omega} F \cdot v dx + \sum_{i=1}^2 \int_{\Gamma} B_i (v \cdot \tau_i) dS \end{aligned} \quad (4.2)$$

is satisfied $\forall v \in V$;

2. (3.1)₂ is satisfied in the following sense:

$$-\int_{\Omega} \sigma \tilde{u} \cdot \nabla \phi \, dx - \int_{\Omega} \sigma \phi \operatorname{div} \tilde{u} \, dx = \int_{\Omega} \phi (G - \operatorname{div} u) \, dx + \int_{\Gamma_{in}} \sigma_{in} \phi \, dS, \quad (4.3)$$

$\forall \phi \in C^1(\overline{\Omega}) : \phi|_{\Gamma_{out}} = 0$, where \tilde{u} is defined in (4.1), and

3. the equation

$$r_0 \int_{\Omega} \partial_{x_1} \eta w + r_1 \int_{\Omega} w \operatorname{div} u + \kappa \int_{\Omega} \nabla w \cdot \nabla \eta + \int_{\Gamma} L w \eta \, dS = \int_{\Omega} H w \, dx \quad (4.4)$$

holds for all $w \in W_2^1$.

We introduce an orthonormal basis of V formed by $\{\omega_i\}_{i=1}^{\infty}$. We consider finite dimensional spaces: $V^N = \{\sum_{i=1}^N \alpha_i \omega_i : \alpha_i \in \mathbb{R}\} \subset V$. The sequence of approximations to the velocity will be searched for as $u^N = \sum_{i=1}^N c_i^N \omega_i$. Due to the equation (3.1)₂ we have to define the approximations to the density in an appropriate way. Namely, we set $\sigma^N = S(G - \operatorname{div} u^N)$, where $S : L_2(\Omega) \rightarrow L_{\infty}(L_2)$ is defined in (2.25), with $w_{in} = \sigma_{in}$.

Finally, to define the approximations of the temperature we introduce $R : H^1 \rightarrow H^1$ as a weak solution operator to equation (3.1)_{3,7}, i.e. we set

$$\eta^N = R(u^N) \iff (4.4) \text{ holds with } \eta := \eta^N \forall w \in W_2^1.$$

The well-posedness of the operator R is direct as it is just a solution operator for a standard elliptic equation. To show it is well defined it is enough to recall the estimate from the proof of (3.4). In the same way we show the estimate for (4.4) which gives existence of η satisfying (4.4) for given $u \in W_2^1$, hence R is well defined. In particular, we have

$$\|R(u)\|_{W_2^1} \leq C[\|F\|_{L_2} + \|u\|_{W_2^1}]. \quad (4.5)$$

With the operators S and R well defined we can proceed with the Galerkin method. Taking $u = u^N = \sum_i c_i^N \omega_i$, $v = \omega_k$, $k = 1 \dots N$ and $\sigma = \sigma^N = S(G - \operatorname{div} u^N)$ in (4.2), $\eta = \eta^N = R(u^N)$, we arrive at a system of N equations

$$B^N(u^N, \omega_k) = 0, \quad k = 1 \dots N, \quad (4.6)$$

where $B^N : V^N \rightarrow V^N$ is defined as

$$\begin{aligned} B^N(\xi^N, v^N) = & \int_{\Omega} \{v^N \partial_{x_1} \xi^N + \mathbf{S}(\xi^N) : \nabla v^N\} \, dx \\ & - p_1 \int_{\Omega} S(G - \operatorname{div} \xi^N) \operatorname{div} v^N \, dx - p_2 \int_{\Omega} R(\xi^N) \operatorname{div} v^N \, dx \\ & + \sum_{j=1}^2 \int_{\Gamma} [\alpha(\xi^N \cdot \tau_j) - B_j] (v^N \cdot \tau_j) \, dS - \int_{\Omega} F \cdot v^N \, dx. \end{aligned} \quad (4.7)$$

Now, if u^N satisfies (4.6) for $k = 1 \dots N$ and σ^N and η^N , defined as above, then the triple (u^N, σ^N, η^N) satisfies (4.2)–(4.4) for $(v, \phi, w) \in (V^N \times C^1(\overline{\Omega})) \times W_2^1$, $\phi|_{\Gamma_{out}} = 0$. We will call such a triple an approximate solution to (4.2)–(4.4).

The following lemma gives existence of a solution to system (4.6):

Lemma 15. Let $F, G \in L^2(\Omega)$, $\sigma_{in} \in L_2(\Gamma_{in})$, $B_i \in L_2(\Gamma)$, $i = 1, 2$. Assume that $\|U\|_{W_p^2}$ and κ, L, α fulfill the assumptions of Theorem 1. Then there exists $u^N \in V^N$ satisfying (4.6) for $k = 1 \dots N$. Moreover,

$$\|u^N\|_{W_2^1} \leq C(DATA). \quad (4.8)$$

Proof. We will apply a well-known tool of the Galerkin method, Lemma 5. Thus we define $P^N : V^N \rightarrow V^N$ as

$$P^N(\xi^N) = \sum_k B^N(\xi^N, \omega_k) \omega_k \quad \text{for } \xi^N \in V^N. \quad (4.9)$$

We have to show that $(P^N(\xi^N), \xi^N) > 0$ on some sphere in V^N . As B is linear in the second variable, we have

$$\begin{aligned} (P(\xi^N), \xi^N) &= B^N(\xi^N, \xi^N) = \underbrace{\int_{\Omega} \mathbf{S}(\nabla \xi^N) : \nabla \xi^N dx}_{I_1} \\ &+ \underbrace{\int_{\Omega} \xi^N \partial_{x_1} \xi^N dx + \int_{\Gamma} \alpha (\xi^N \cdot \tau_i)^2 dS - p_1 \int_{\Omega} S(G - \operatorname{div} \xi^N) \operatorname{div} \xi^N dx}_{I_2} \\ &\underbrace{- p_2 \int_{\Omega} R(\xi^N) \operatorname{div} v^N dS - \int_{\Omega} F \cdot \xi^N dx - \sum_{i=1}^2 \int_{\Gamma} B_i (\xi^N \cdot \tau_i) dS}_{I_4}. \end{aligned} \quad (4.10)$$

By the Korn inequality we have

$$I_1 + I_2 \geq C \|\xi^N\|_{W_2^1}^2 \quad (4.11)$$

for α large enough. To deal with I_4 let us denote for a moment $\eta^N = R(\xi^N)$. Then setting $w = \eta^N$ in (4.4) (with ξ^N instead of u) we have

$$I_4 = -p_2 \int_{\Omega} \eta^N \operatorname{div} \xi^N = \frac{p_2}{r_1} \left[\int_{\Omega} \partial_{x_1} \eta^N \eta^N dx + \kappa \int_{\Omega} |\nabla \eta^N|^2 dx + \int_{\Gamma} L(\eta^N)^2 dS - \int_{\Omega} H \eta^N dx \right]$$

and repeating the reasoning from the proof of (3.4) we infer

$$I_4 \geq -\|H^N\|_{L_2} \|\eta^N\|_{L^2} \geq -\|H^N\|_{L_2} (\|H^N\|_{L_2} + \|\xi^N\|_{W_2^1}). \quad (4.12)$$

We have to find a bound on I_3 . Denoting $\sigma^N = S(G - \operatorname{div} \xi^N)$ we have

$$- \int_{\Omega} \sigma^N \operatorname{div} \xi^N dx = \int_{\Omega} \sigma^N (\partial_{x_1} \sigma^N + U \cdot \nabla \sigma^N) dx - \int_{\Omega} \sigma^N G dx. \quad (4.13)$$

Using (2.26) we get

$$- \int_{\Omega} \eta^N G^N dx \geq -\|\eta^N\|_{L^2} \|G^N\|_{L^2} \geq -C \|G^N\|_{L_2} (\|G^N\|_{L_2} + \|\xi^N\|_{W_2^1} + \|\sigma_{in}\|_{L_2(\Gamma_{in})}). \quad (4.14)$$

The remaining part of (4.13) is also not very difficult. With the first integral on the r.h.s we have

$$\begin{aligned}
& \int_{\Omega} \sigma^N (\partial_{x_1} \sigma^N + U \cdot \nabla \sigma^N) dx = \frac{1}{2} \int_{\Omega} (\partial_{x_1} |\sigma^N|^2 + U \cdot \nabla |\sigma^N|^2) dx \\
& = -\frac{1}{2} \int_{\Gamma_{in}} |\sigma^N|^2 dS + \frac{1}{2} \int_{\Gamma_{out}} |\sigma^N|^2 dS + \frac{1}{2} \int_{\Gamma} U \cdot n |\sigma^N|^2 dS - \frac{1}{2} \int_{\Omega} |\sigma^N|^2 \operatorname{div} U dx \\
& \geq -\frac{1}{2} \int_{\Gamma_{in}} \sigma_{in}^2 dS + \frac{1}{2} \int_{\Gamma} U \cdot n |\sigma_{in}|^2 dS - \frac{1}{2} \|\operatorname{div} U\|_{L_{\infty}} \|\sigma^N\|_{L_2}^2 \\
& \geq -C \|\sigma_{in}\|_{L_2}^2 - E \|\sigma^N\|_{L_2}^2 \geq -C - E (\|G\|_{L_2}^2 + \|\sigma^N\|_{W_2^1}^2 + \|\sigma_{in}\|_{L_2}^2).
\end{aligned} \tag{4.15}$$

Hence we have

$$I_3 \geq -C (\|G\|_{L_2}^2 + \|\sigma_{in}\|_{L_2(\Gamma_{in})}^2) - E \|\xi^N\|_{W_2^1}^2. \tag{4.16}$$

Combining (4.11), (4.12) and (4.16) we conclude

$$(P^N(\xi^N), \xi^N) \geq C [\|\xi^N\|_{W_2^1}^2 - D \|\xi^N\|_{W_2^1} - D^2], \tag{4.17}$$

where $D = \|F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} + \|\sigma_{in}\|_{L_2(\Gamma_{in})} + \sum_{i=1}^2 \|B_i\|_{L_2(\Gamma)}$. Thus there exists $M = M(\mu, \Omega, D)$ such that $(P^N(\xi^N), \xi^N) > 0$ for $\|\xi^N\|_{W_2^1} = M$, and applying Lemma 5 we conclude that $\exists \xi^{N*} : P^N(\xi^{N*}) = 0$ and $\|\xi^{N*}\|_{W_2^1} \leq M$. By the definition of P^N , $u^N = \xi^{N*}$ is a solution to (4.6). \square

Now showing the existence of weak solution is straightforward. The result is

Lemma 16. *Assume that $F, G, H \in L_2(\Omega)$, $\sigma_{in} \in L_2(\Gamma_{in})$, $B_i \in L_2(\Gamma)$, $i = 1, 2$. Let $\|U\|_{W_p^2}$ be small enough and assume κ, L, f satisfy the assumptions of Theorem 1. Then there exists $(u, \sigma, \eta) \in V \times L_{\infty}(L_2) \times W_2^1$, that is a weak solution to system (3.1). Moreover, the weak solution satisfies estimate (3.4).*

Proof. Let us set $\sigma^N = S(G - \operatorname{div} u^N)$ and $\eta^N = R(u^N)$, where u^N is the solution to (4.6). Estimates (2.26), (4.5) and (4.8) imply that $\|u^N\|_{H^1} + \|\sigma^N\|_{L_{\infty}(L_2)} + \|\eta^N\|_{H^1} \leq C(D_{lin})$. Thus, at least for a chosen subsequence (denoted however in the same way)

$$u^N \rightharpoonup u \quad \text{in } H^1, \quad \sigma^N \rightharpoonup^* \sigma \quad \text{in } L_{\infty}(L_2) \quad \text{and} \quad \eta^N \rightharpoonup \eta \quad \text{in } W_2^1$$

for some $(u, \sigma, \eta) \in H^1 \times L_{\infty}(L_2) \times H^1$. Passing to the limit in (4.2)–(4.4) for (u^N, σ^N, η^N) we conclude that (u, σ, η) satisfies (4.2)–(4.4), thus we have the weak solution. To show the boundary condition on the density we can rewrite the r.h.s of (3.1)_{2,6} as

$$\begin{cases} \partial_{x_1} \sigma + \frac{U^2}{1+U^1} \partial_{x_2} \sigma + \frac{U^3}{1+U^1} \partial_{x_3} \sigma = \frac{G - \operatorname{div} u}{1+U^1} & \text{in } \mathcal{D}'(\Omega), \\ \sigma = \sigma_{in} & \text{on } \Gamma_{in}, \end{cases} \tag{4.18}$$

and, treating x_1 as a "time" variable, adapt Di Perna-Lions theory of transport equation ([5]) that implies the uniqueness of solution to (4.18) in the class $L_{\infty}(L_2)$. The proof is thus complete. \square

4.2 Strong solution

With the estimate (3.11) the only problem is to tackle the singularities at the junction of Γ_0 with Γ_{in} and Γ_{out} . To this end we reflect the weak solution in a way which preserves the boundary conditions and then apply the classical elliptic theory to the extended solution of the Lamé system. The details are given in Lemma 7. To show the regularity of the temperature we can apply Lemma 6 since on $\Gamma_{in} \cup \Gamma_{out}$ the boundary condition on the density reduces to the Neumann condition.

5 Bounds on the approximating sequence

In this section we will show the bounds on the sequence (2.24). Due to the term $u \cdot \nabla \sigma$ in the continuity equation we are not able to show directly the convergence of this sequence in $W_p^2 \times W_p^1 \times W_p^2$ to the strong solution of (2.20). We can show however its boundedness in $W_p^2 \times W_p^1 \times W_p^2$. Next, using this bound we derive the Cauchy condition in $W_2^1 \times L_\infty(L_2) \times W_2^1$, and thus show the convergence in this space to some (u, σ, η) . On the other hand, the boundedness implies weak convergence in $W_p^2 \times W_p^1 \times W_p^2$, and the limit must be (u, σ, η) ; hence the solution is strong.

The following lemma gives the boundedness of (u^n, σ^n, η^n) in $W_p^2 \times W_p^1 \times W_p^2$.

Lemma 17. *Let $\{(u^n, \sigma^n, \eta^n)\}$ be a sequence of solutions to (2.24) starting from $(u^0, \sigma^0, \eta^0) = ([0, 0, 0], 0, 0)$. Then*

$$\|u^n\|_{W_p^2} + \|\sigma^n\|_{W_p^1} + \|\eta^n\|_{W_p^2} \leq M, \quad (5.1)$$

where M can be arbitrarily small provided that D_0 defined in (1.14), quantities $\|B_i\|_{W_p^{1-1/p}(\Gamma)}$, $i = 1, 2$, $\|\sigma_{in} - 1\|_{W_p^1(\Gamma_{in})}$ and $\|U\|_{W_p^2}$ are small enough and α, L, κ fulfill the assumptions of Theorem 1.

Proof. Estimate (3.11) for system (2.24) reads

$$\begin{aligned} & \|u^{n+1}\|_{W_p^2} + \|\sigma^{n+1}\|_{W_p^1} + \|\eta^{n+1}\|_{W_p^2} \leq \\ & \leq C [\|F(u^n, \sigma^n, \eta^n)\|_{L_p} + \|G(u^n, \sigma^n, \eta^n)\|_{W_p^1} + \|H(u^n, \sigma^n, \eta^n)\|_{L_p} \\ & \quad + \sum_{i=1}^2 \|B_i\|_{W_p^{1-1/p}(\Gamma)} + \|\sigma_{in}\|_{W_p^1(\Gamma_{in})}]. \end{aligned} \quad (5.2)$$

Denoting $A_n = \|u^n\|_{W_p^2} + \|\sigma^n\|_{W_p^1} + \|\eta^n\|_{W_p^2}$, from (2.23) and (5.2) we get

$$A_{n+1} \leq C(A_n^2 + A_n^3 + D_0). \quad (5.3)$$

We aim at showing that A_n thus A_n is bounded by a constant that can be arbitrarily small provided that A_0 and D_0 are small enough. Indeed, let us fix $0 < \delta < \frac{1}{8C}$. (Note that the constant C can be without loss of generality taken larger than 1.) Assume that $CD_0 < \delta$. Then (5.3) entails an implication $A_n \leq 2\delta \Rightarrow A_{n+1} \leq 2\delta$ and we can conclude that

$$\|u^n\|_{W_p^2} + \|\sigma^n\|_{W_p^1} + \|\eta^n\|_{W_p^2} \leq 2\delta \quad \forall n \in \mathbb{N}. \quad (5.4)$$

□

The next lemma almost completes the proof of the Cauchy condition in $W_2^1 \times L_\infty(L_2) \times W_2^1$ for the iterating scheme.

Lemma 18. *Let the assumptions of Lemma 17 hold. Then we have*

$$\begin{aligned} & \|u^{n+1} - u^{m+1}\|_{W_2^1} + \|\sigma^{n+1} - \sigma^{m+1}\|_{L_\infty(L_2)} + \|\eta^{n+1} - \eta^{m+1}\|_{W_2^1} \\ & \leq E(M) (\|u^n - u^m\|_{W_2^1} + \|\sigma^n - \sigma^m\|_{L_\infty(L_2)} + \|\eta^n - \eta^m\|_{W_2^1}), \end{aligned} \quad (5.5)$$

where M is the constant from (5.1) and $E(M)$ can be taken arbitrarily small.

Proof. Subtracting $(2.24)_m$ from $(2.24)_n$ we arrive at

$$\begin{aligned} & \partial_{x_1}(u^{n+1} - u^{m+1}) - \mathbf{S}(\nabla(u^{n+1} - u^{m+1})) + p_1 \nabla(\sigma^{n+1} - \sigma^{m+1}) \\ & + p_2 \nabla(\eta^{n+1} - \eta^{m+1}) = F(u^n, \sigma^n, \eta^n) - F(u^m, \sigma^m, \eta^m), \end{aligned} \quad (5.6)$$

$$\begin{aligned} & \operatorname{div}(u^{n+1} - u^{m+1}) + \partial_{x_1}(\sigma^{n+1} - \sigma^{m+1}) + (u^n + u_0) \cdot \nabla(\sigma^{n+1} - \sigma^{m+1}) = \\ & = G(u^n, \sigma^n) - G(u^m, \sigma^m) - (u^n - u^m) \cdot \nabla \sigma^{m+1}, \end{aligned} \quad (5.7)$$

$$\begin{aligned} & r_1 \partial_{x_1}(\eta^{n+1} - \eta^{m+1}) + r_2 \operatorname{div}(u^{n+1} - u^{m+1}) - \kappa \Delta(\eta^{n+1} - \eta^{m+1}) = \\ & = H(u^n, \sigma^n, \eta^n) - H(u^m, \sigma^m, \eta^m) \end{aligned} \quad (5.8)$$

$$\begin{aligned} & \mathbf{S}(\nabla(u^{n+1} - u^{m+1})) \cdot \tau_i + \alpha (u^{n+1} - u^{m+1}) \cdot \tau_i|_\Gamma = 0, \\ & n \cdot (u^{n+1} - u^{m+1})|_\Gamma = 0, \\ & \sigma^{n+1} - \sigma^{m+1}|_{\Gamma_{in}} = 0, \\ & \kappa \frac{\partial(\eta^{n+1} - \eta^{m+1})}{\partial n} + L(\eta^{n+1} - \eta^{m+1}) = 0. \end{aligned} \quad (5.9)$$

Estimate (3.4) applied to this system yields

$$\begin{aligned} & \|u^{n+1} - u^{m+1}\|_{W_2^1} + \|\sigma^{n+1} - \sigma^{m+1}\|_{L_\infty(L_2)} + \|\eta^{n+1} - \eta^{m+1}\|_{W_2^1} \leq \\ & \|F(u^n, \sigma^n, \eta^n) - F(u^m, \sigma^m, \eta^m)\|_{V^*} + \|G(u^n, \sigma^n) - G(u^m, \sigma^m)\|_{L_2} \\ & + \|(u^n - u^m) \cdot \nabla \sigma^{m+1}\|_{L_2} + \|H(u^n, \sigma^n, \eta^n) - H(u^m, \sigma^m, \eta^m)\|_{L_2}. \end{aligned}$$

In order to derive (5.5) from the above inequality we have to examine the r.h.s. The differences in G and H are bounded in a straightforward way: using the imbedding $W_p^1 \hookrightarrow L_\infty$ and Hölder's inequality we derive

$$\begin{aligned} & \|G(u^n, \sigma^n) - G(u^m, \sigma^m)\|_{L_2} + \|H(u^n, \sigma^n, \eta^n) - H(u^m, \sigma^m, \eta^m)\|_{L_2} \\ & \leq E(M) (\|u^n - u^m\|_{W_2^1} + \|\sigma^n - \sigma^m\|_{L_\infty(L_2)} + \|\eta^n - \eta^m\|_{W_2^1}). \end{aligned} \quad (5.10)$$

The difference in F must be investigated more carefully. A direct calculation yields

$$F(u^n, \sigma^n, \eta^n) - F(u^m, \sigma^m, \eta^m) = F_1^{n,m} + F_2^{n,m},$$

where in $F_1^{n,m}$ we include all the terms which do not contain $\nabla(\sigma^n - \sigma^m)$ as well as other terms containing the gradient of density or its difference. Direct calculation using the imbedding $H^1 \hookrightarrow L_6$, $W_p^1 \hookrightarrow L_\infty$ and Hölder's inequality yields

$$\|F_1^{n,m}\|_{V^*} \leq E(M) (\|u^n - u^m\|_{W_2^1} + \|\sigma^n - \sigma^m\|_{L_\infty(L_2)}). \quad (5.11)$$

Next,

$$\begin{aligned} F_2^{n,m} &= -(\partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) - p_1) \nabla(\sigma^n - \sigma^m) \\ &\quad - (\partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) - \partial_\rho \pi(\sigma^m + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^m)) \nabla \sigma^m \\ &=: F_{2,1}^{n,m} + F_{2,2}^{n,m}. \end{aligned} \quad (5.12)$$

We have to compute V^* norm of $F_2^{n,m}$, hence we multiply by $v \in V$ and integrate. With the first term we have

$$\begin{aligned} &\left| \int_\Omega -(\partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) - p_1) \nabla(\sigma^n - \sigma^m) \cdot v \, dx \right| \\ &\leq \left| \int_\Omega (\partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) - p_1) (\sigma^n - \sigma^m) \operatorname{div} v \, dx \right| \\ &\quad + \left| \int_\Omega \nabla \partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) \cdot v (\sigma^n - \sigma^m) \, dx \right| \\ &\leq C(\|\eta^n\|_{W_p^1} + \|\sigma^n\|_{W_p^1} + \|\bar{\theta}_0 - T_0\|_{W_p^1} + \|\bar{\theta}_1\|_{W_p^1}) \|(\sigma^n - \sigma^m)\|_{L_2} \|v\|_{W_2^1}, \end{aligned}$$

hence

$$\|F_{2,1}^{n,m}\|_{V^*} \leq E(M) \|(\sigma^n - \sigma^m)\|_{L_\infty(L_2)}. \quad (5.13)$$

We have used the fact that $\frac{1}{2} + \frac{1}{6} + \frac{1}{p} < 1$ and the imbedding $W_2^1 \hookrightarrow L^6$. The other term $F_{2,2}^{n,m}$ can be estimated similarly, only without integration by parts. Combining the estimates on $F_1^{n,m}, F_{2,1}^{n,m}$ and $F_{2,2}^{n,m}$ we conclude

$$\|F(u^n, \sigma^n, \eta^n) - F(u^m, \sigma^m, \eta^m)\|_{V^*} \leq E(M) (\|u^n - u^m\|_{W_2^1} + \|\sigma^n - \sigma^m\|_{L_\infty(L_2)} + \|\eta^n - \eta^m\|_{W_2^1}). \quad (5.14)$$

The only term that remains to estimate is $(u^n - u^m) \cdot \nabla \sigma^m$. We emphasize that this is the term which makes it impossible to show the convergence in $W_p^2 \times W_p^1$ directly. Namely, if we would like to apply the estimate (3.11) to the system for the difference then we would have to estimate $\|(u^n - u^m) \cdot \nabla \sigma^m\|_{W_p^1}$ what can not be done as we do not have any knowledge about $\|\sigma^m\|_{W_p^2}$. Fortunately we only need the L_2 -norm of this awkward term, which can be bounded in a direct way as

$$\|(u^n - u^m) \cdot \nabla \sigma^m\|_{L_2} \leq \|u^n - u^m\|_{L_q} \|\nabla \sigma^m\|_{L_p} \leq C \|\sigma^m\|_{W_p^1} \|u^n - u^m\|_{W_2^1}, \quad (5.15)$$

since $q = \frac{2p}{p-2} < 6$ for $p < 3$. We have thus completed the proof of (5.5). \square

Now, Lemma 17 implies that the constant $E(M) < 1$ provided that the data is small enough and the starting point $(u^0, \sigma^0, \eta^0) = ([0, 0, 0], 0, 0)$. It completes the proof of the Cauchy condition in $H^1 \times L_\infty(L_2) \times H^1$ for the sequence (u^n, σ^n, η^n) .

Remark. Lemmas 17 and 18 hold for any starting point (u^0, σ^0, η^0) small enough in $W_p^2 \times W_p^1 \times W_p^2$, but we can start the iteration from $([0, 0, 0], 0, 0)$ without loss of generality.

6 Proof of Theorem 1

In this section we prove our main result, Theorem 1. First we show existence of the solution passing to the limit with the sequence (u^n, σ^n, η^n) and next we show that this solution is unique in the class of solutions satisfying (1.15).

Existence of the solution. Since we have the Cauchy condition on the sequence (u^n, σ^n, η^n) only in the space $W_2^1 \times L_\infty(L_2) \times W_2^1$, first we have to show the convergence in the weak formulation of the problem (2.20). The sequence (u^n, σ^n, η^n) satisfies in particular the following weak formulation of (2.24):

$$\begin{aligned} & \int_{\Omega} \{v \cdot \partial_{x_1} u^{n+1} + \mathbf{S}(\nabla u^{n+1}) : \nabla v - p_1 \sigma^{n+1} \operatorname{div} v - p_2 \eta^{n+1} \operatorname{div} v\} dx \\ & + \int_{\Gamma} \alpha(u^{n+1} \cdot \tau_i) (v \cdot \tau_i) dS = \int_{\Omega} F(u^n, \sigma^n, \eta^n) \cdot v dx + \sum_{i=1}^2 \int_{\Gamma} B_i(v \cdot \tau_i) dS \quad \forall v \in V \end{aligned} \quad (6.1)$$

$$- \int_{\Omega} \sigma^{n+1} \tilde{u}^n \cdot \nabla \phi dx - \int_{\Omega} \sigma^{n+1} \phi \operatorname{div} \tilde{u}^{n+1} dx = \int_{\Omega} \phi (G(u^n, \sigma^n) - \operatorname{div} u^{n+1}) dx + \int_{\Gamma_{in}} \sigma_{in} \phi dS \quad (6.2)$$

$\forall \phi \in C^\infty(\bar{\Omega}) : \phi|_{\Gamma_{out}} = 0$, where $\tilde{u}^n = [1 + (u^n + u_0)^1, (u^n + u_0)^2, (u^n + u_0)^3]$, and

$$\begin{aligned} & (1 + e_2) \int_{\Omega} \partial_{x_1} \eta^{n+1} w + T_0 p_2 \int_{\Omega} w \operatorname{div} u^{n+1} + \kappa \int_{\Omega} \nabla w \cdot \nabla \eta^{n+1} \\ & + \int_{\Gamma} L w \eta^{n+1} dS = \int_{\Omega} H(u^n, \sigma^n, \eta^n) w dx \end{aligned} \quad (6.3)$$

$\forall w \in H^1$.

Now using the convergence in $W_2^1 \times L_\infty(L_2) \times W_2^1$ combined with the bound (5.1) in $W_p^2 \times W_p^1 \times W_p^2$ we can pass to the limit in (6.1)–(6.3). The convergence of the l.h.s. of (6.1)–(6.3) is obvious. Recalling the definition (2.21) of $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$ we verify easily that

$$\int_{\Omega} G(u^n, \sigma^n) \cdot v dx \rightarrow \int_{\Omega} G(u, \sigma) \cdot v dx \quad (6.4)$$

and

$$\int_{\Omega} H(u^n, \sigma^n, \eta^n) \cdot v dx \rightarrow \int_{\Omega} H(u, \sigma, \eta) \cdot v dx, \quad (6.5)$$

and the only step in showing the convergence of F which requires more attention is to show that

$$\int_{\Omega} (\partial_\rho \pi(\sigma^n + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta^n) - p_1) \nabla \sigma^n \cdot v dx \rightarrow \int_{\Omega} (\partial_\rho \pi(\sigma + 1, \bar{\theta}_0 + \bar{\theta}_1 + \eta) - p_1) \nabla \sigma \cdot v dx. \quad (6.6)$$

However, since $\sigma^n \rightarrow \sigma$ and $\eta^n \rightarrow \eta$ in $C(\bar{\Omega})$ as well as $\nabla \sigma^n \rightharpoonup \nabla \sigma$ in $L_p(\Omega)$, we easily verify that (6.6) holds true. We conclude that (u, σ, η) satisfies (6.1)–(6.3). Now we need to show that this implies the strong formulation (2.20), what can be done in a standard way, just integrating by parts in the weak formulation.

Now we set

$$v = \bar{v} + u + u_0, \quad \rho = \sigma + 1, \quad \theta = \eta + \bar{\theta}_0 + \bar{\theta}_1$$

and we see that (v, ρ, θ) solves (1.1). We clearly have $E(M) = E(D_0)$ where D_0 is defined in (1.14), hence estimate (1.15) holds.

Uniqueness. The uniqueness in the class of solutions satisfying (1.15) actually results directly from the method of the proof, more precisely from the proof of (5.5). Namely, we can

show uniqueness on the level of perturbations, i.e. uniqueness for (2.20). Consider two solutions with the same data, denote it by (u_1, σ_1, η_1) and (u_2, σ_2, η_2) . Their difference then satisfy

$$\begin{aligned}
& \partial_{x_1}(u_1 - u_2) - \mathbf{S}(\nabla u_1 - \nabla u_2) + p_1 \nabla(\sigma_1 - \sigma_2) \\
& + p_2 \nabla(\eta_1 - \eta_2) = F(u_1, \sigma_1, \eta_1) - F(u_2, \sigma_2, \eta_2), \\
& \operatorname{div}(u_1 - u_2) + \partial_{x_1}(\sigma_1 - \sigma_2) + (u_1 + u_0) \cdot \nabla(\sigma_1 - \sigma_2) \\
& = G(u_1, \sigma_1) - G(u_2, \sigma_2) - (u_1 - u_2) \cdot \nabla \sigma_2, \\
& (1 + e_2) \partial_{x_1}(\eta_1 - \eta_2) + T_0 p_2 \operatorname{div}(u_1 - u_2) - \kappa \Delta(\eta_1 - \eta_2) \\
& = H(u_1, \sigma_1, \eta_1) - H(u_2, \sigma_2, \eta_2) \\
& \mathbf{S}(\nabla(u_1 - u_2)n) \cdot \tau_i + \alpha(u_1 - u_2) \cdot \tau_i|_{\Gamma} = 0, \\
& n \cdot (u_1 - u_2)|_{\Gamma} = 0, \\
& \sigma_1 - \sigma_2|_{\Gamma_{in}} = 0, \\
& \kappa \frac{\partial(\eta_1 - \eta_2)}{\partial n} + L(\eta_1 - \eta_2) = 0.
\end{aligned}$$

Now recall that to show (5.5) we applied only (5.6)–(5.9) and (5.1), hence from the above equations we conclude

$$\begin{aligned}
& \|u_1 - u_2\|_{W_2^1} + \|\sigma_1 - \sigma_2\|_{L_\infty(L_2)} + \|\eta_1 - \eta_2\|_{W_2^1} \\
& \leq E(M) (\|u_1 - u_2\|_{W_2^1} + \|\sigma_1 - \sigma_2\|_{L_\infty(L_2)} + \|\eta_1 - \eta_2\|_{W_2^1}).
\end{aligned} \tag{6.7}$$

Provided the data are small enough we have $E(M) < 1$ and so

$$\|u_1 - u_2\|_{W_2^1} + \|\sigma_1 - \sigma_2\|_{L_\infty(L_2)} + \|\eta_1 - \eta_2\|_{W_2^1} = 0$$

which completes the proof of uniqueness and hence of Theorem 1.

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